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On an initial-value problem for second order partial differential equations with self-reference

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Abstract. In this paper, we study the local existence and uniqueness of the solution to an initial-value problem for a second-order partial differential equation with self-reference.

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1 Introduction

In [1], Eder obtained the existence, uniqueness, analyticity and analytic dependence of solutions to the following equation of an one-variable unknown function $u : I \subset \mathbb{R} \rightarrow \mathbb{R}$:

$$u'(t) = u(u(t)). \quad (1.1)$$

This is so-called a differential equation with self-reference, since the right-hand side is the composition of the unknown and itself. This equation has attracted much attention. As a more general case than (1.1), Si and Cheng [4] investigated the functional-differential equation

$$u'(t) = u(at + bu(t)), \quad (1.2)$$

where $a \neq 1$ and $b \neq 0$ are complex numbers; the unknown $u : \mathbb{C} \rightarrow \mathbb{C}$ is a complex function. By using the power series method, analytic solutions of this equation are obtained. By generalizing (1.2), in [9] Cheng, Si and Wang considered the equation

$$\alpha t + \beta u'(t) = u(at + bu'(t)),$$

where α and β are complex numbers. Existence theorems are established for the analytic solutions, and systematic methods for deriving explicit solutions are also given.

In [11], Stanek studied maximal solutions of the functional-differential equation

$$u(t)u'(t) = ku(u(t)) \quad (1.3)$$

with $0 < |k| < 1$. Here $u : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a real unknown. This author showed that properties of maximal solutions depend on the sign of the parameter k for two separate cases $k \in (-1, 0)$ and $k \in (0, 1)$. For earlier work of Stanek than (1.3), see [16]–[21].

For a more general model than the above, in [6], Miranda and Pascali studied the existence and uniqueness of a local solution to the following initial-valued problem for a partial differential equation with self-reference and heredity

$$\begin{cases} \frac{\partial}{\partial t}u(x, t) = u \left(\int_0^t u(x, s)ds, t \right), & x \in \mathbb{R}, \text{ a.e. } t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.4)$$

by assuming that u_0 is a bounded, Lipschitz continuous function. With suitable weaker conditions on u_0 , namely u_0 is a non-negative, non-decreasing, bounded, lower semi-continuous real function, in [3], Pascali and Le obtained the existence of a global solution of (1.4).

In [22], T. Nguyen and L. Nguyen, generalizing [7], studied the system of partial differential equations with self-reference and heredity

$$\begin{cases} \frac{\partial}{\partial t}u(x, t) = u \left(\alpha v(x, t) + v \left(\int_0^t u(x, s)ds, t \right) t \right), \\ \frac{\partial}{\partial t}v(x, t) = v \left(\beta u(x, t) + u \left(\int_0^t v(x, s)ds, t \right) t \right), \end{cases} \quad (1.5)$$

associated with initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad (1.6)$$

where α and β are non-negative coefficients. By the boundedness and Lipschitz continuity of u_0 and v_0 , we obtained the existence and uniqueness of a local solution to this system. We also proved that this system has a global solution, provided u_0 and v_0 are non-negative, non-decreasing, bounded and lower semi-continuous functions.

In [5], Pascali and Miranda considered an initial-valued problem for a second-order partial differential equation with self-reference as follows:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x, t) = k_1 u \left(\frac{\partial^2}{\partial t^2}u(x, t) + k_2 u(x, t), t \right), \\ u(x, 0) = \alpha(x), \\ \frac{\partial}{\partial t}u(x, 0) = \beta(x). \end{cases} \quad (1.7)$$

These authors proved that if $\alpha(x)$ and $\beta(x)$ are bounded and Lipschitz continuous functions, k_1 and k_2 are given real numbers, this problem has a unique local solution. It is noted that this result still holds when $k_i \equiv k_i(x, t)$, $i = 1, 2$, are real functions satisfying some technical conditions.

Motivated from problem (1.7) and related questions in [5], in this paper we establish the existence and uniqueness of a local solution to the following Cauchy problem of an partial differential equation with self-reference:

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x, t) = \mu_1 u \left(\frac{\partial^2}{\partial t^2}u(x, t) + \mu_2 u \left(\frac{\partial^2}{\partial t^2}u(x, t) + \mu_3 u(x, t), t \right), t \right), \\ u(x, 0) = p(x) \\ \frac{\partial}{\partial t}u(x, 0) = q(x), \end{cases} \quad (1.8)$$

where p and q are given functions, μ_i , $i = 1, 2, 3$, given real numbers $x \in \mathbb{R}$ and $t \in [0, T]$ for some $T > 0$. It is clear that this problem is a non-trivial generalization of (1.7). Let us specify some reasons as follows:

- The operator

$$\frac{\partial^2}{\partial t^2}u(x, t) + \mu_2 u \left(\frac{\partial^2}{\partial t^2}u(x, t) + \mu_3 u(x, t), t \right)$$

is actually a doubly self-reference form, which is more complicated than that of (1.7);

- If $k_2 = \mu_2 = 0$, problem (1.8) coincides with problem (1.7). This is the only coincidence of these two problems. This means that the problem we study in this paper is not a “natural” generalization of (1.7), not including (1.7) as a special case.

Finally we present the problem (1.8) in the case that $p(x) = p_0$ and $q(x) = q_0$, where p_0 and q_0 are two given constants and we remark a particular strange situation.

2 Existence and uniqueness of a local solution

By integrating the partial differential equation in (1.8), we obtain the following integral equation:

$$u(x, t) = u_0(x, t) + \int_0^t \int_0^\tau \mu_1 u \left(\frac{\partial^2}{\partial s^2} u(x, s) + \mu_2 u \left(\frac{\partial^2}{\partial s^2} u(x, s) + \mu_3 u(x, s), s \right), s \right) ds d\tau, \quad (2.9)$$

where $u_0(x, t) = p(x) + tq(x)$ and $x \in \mathbb{R}$ and $t \in [0, T]$.

The following theorem is so clear that its proof is omitted.

Theorem 2.1. If u is a continuous solution of problem (2.9), then it is also a solution of problem (1.8).

This theorem allows us to consider problem (2.9) only in the rest of this paper. For simplicity, we assume that $|\mu_1| = |\mu_2| = |\mu_3| = 1$. Now we state our main result.

Theorem 2.2. Assume that p and q are bounded and Lipschitz continuous on \mathbb{R} . Let σ be the lipschitz constant of p and assume that $\sigma < 1$. Then there exists a positive constant T_0 such that problem (2.9) has a unique solution, denoted by $u_\infty(x, t)$, in $\mathbb{R} \times [0, T_0]$. Moreover, the function $u_\infty(x, t)$ is also bounded and Lipschitz continuous with respect to each of variables $x \in \mathbb{R}$ and $t \in [0, T_0]$.

Proof. To prove this theorem, we use an iterative algorithm. The proof includes some steps as below.

Step 1: An iterate sequence of functions. We define the following sequence of real functions $(u_n)_n$ defined for $x \in \mathbb{R}, t \in [0, T]$ for $T > 0$:

$$\begin{aligned} u_0(x, t) &= p(x) + tq(x), \\ u_1(x, t) &= u_0(x, t) + \int_0^t \int_0^\tau \mu_1 u_0 \left(\mu_2 u_0(\mu_3 u_0(x, s), s), s \right) ds d\tau, \\ u_{n+1}(x, t) &= u_0(x, t) + \int_0^t \int_0^\tau \mu_1 u_n \left(\frac{\partial^2}{\partial s^2} u_n(x, s) + \mu_2 u_n \left(\frac{\partial^2}{\partial s^2} u_n(x, s) \right. \right. \\ &\quad \left. \left. + \mu_3 u_n(x, s), s \right), s \right) ds d\tau. \end{aligned} \quad (2.10)$$

Step 2: Proof of the boundedness of (u_n) . With simple calculations, taking into account the boundedness of p and q , we get

$$|u_0(x, t)| \leq |p(x)| + t|q(x)| \leq \|p\|_{L^\infty} + t\|q\|_{L^\infty},$$

$$\begin{aligned}
|u_1(x, t)| &\leq |u_0(x, t)| + \int_0^t \int_0^\tau \left| \mu_1 u_0(\mu_2 u_0(\mu_3 u_0(x, s), s), s) \right| ds d\tau \\
&\leq \|p\|_{L^\infty} + t\|q\|_{L^\infty} + \int_0^t \int_0^\tau \left(\|p\|_{L^\infty} + s\|q\|_{L^\infty} \right) ds d\tau \\
&= \left(1 + \frac{t^2}{2!} \right) \|p\|_{L^\infty} + \left(t + \frac{t^3}{3!} \right) \|q\|_{L^\infty}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|u_2(x, t)| &\leq |u_0(x, t)| + \int_0^t \int_0^\tau \left| \mu_1 u_1 \left(\frac{\partial^2}{\partial s^2} u_1(x, s) + \mu_2 u_1 \left(\frac{\partial^2}{\partial s^2} u_1(x, s) \right. \right. \right. \\
&\quad \left. \left. \left. + \mu_3 u_1(x, s), s \right), s \right) \right| ds d\tau \\
&\leq \|p\|_{L^\infty} + t\|q\|_{L^\infty} + \int_0^t \int_0^\tau \left(1 + \frac{s^2}{2!} \right) \|p\|_{L^\infty} + \left(s + \frac{s^3}{3!} \right) \|q\|_{L^\infty} ds d\tau \\
&= \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} \right) \|p\|_{L^\infty} + \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} \right) \|q\|_{L^\infty}.
\end{aligned}$$

By induction on n we find

$$|u_n(x, t)| \leq e^T \left(\|p\|_{L^\infty} + \|q\|_{L^\infty} \right), \quad n \in \mathbb{N}, \quad t \in [0, T]. \quad (2.11)$$

Step 3: Every u_n is lipschitz with respect to the first variable. From the Lipschitz continuity of p and q

$$\begin{aligned}
|p(x) - p(y)| &\leq \sigma |x - y|, \quad \forall x, y \in \mathbb{R}, \\
|q(x) - q(y)| &\leq \omega |x - y|, \quad \forall x, y \in \mathbb{R}.
\end{aligned} \quad (2.12)$$

where $0 < \sigma, \omega$ are real numbers (with $\sigma < 1$ as in the hypotheses).

Using (2.12), we derive

$$\begin{aligned}
|u_0(x, t) - u_0(y, t)| &\leq |p(x) - p(y)| + t|q(x) - q(y)| \\
&\leq \left(\sigma + t\omega \right) |x - y| := L_0(t) |x - y|,
\end{aligned} \quad (2.13)$$

where $L_0(t) := \sigma + t\omega$.

In addition,

$$\begin{aligned}
|u_1(x, t) - u_1(y, t)| &\leq L_0(t)|x - y| + \int_0^t \int_0^\tau \left| \mu_1 u_0(\mu_2 u_0(\mu_3 u_0(x, s), s), s) \right. \\
&\quad \left. - \mu_1 u_0(\mu_2 u_0(\mu_3 u_0(y, s), s), s) \right| ds d\tau \\
&\leq \left(L_0(t) + \int_0^t \int_0^\tau L_0^3(s) ds d\tau \right) |x - y| \\
&:= L_1(t)|x - y|,
\end{aligned} \tag{2.14}$$

where $L_1(t) := L_0(t) + \int_0^t \int_0^\tau C_0(s) ds d\tau$, with $C_0(t) := L_0^3(t)$.

Moreover

$$\begin{aligned}
\left| \frac{\partial^2}{\partial t^2} u_1(x, t) - \frac{\partial^2}{\partial t^2} u_1(y, t) \right| &\leq L_0(t) \left| \mu_2 u_0(\mu_3 u_0(x, t), t) - \mu_2 u_0(\mu_3 u_0(y, t), t) \right| \\
&\leq L_0^2(t) \left| \mu_3 u_0(x, t) - \mu_3 u_0(y, t) \right| \\
&\leq L_0^3(t) |x - y| := C_0(t) |x - y|.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&|u_2(x, t) - u_2(y, t)| \\
&\leq L_0(t)|x - y| + \int_0^t \int_0^\tau L_1(s) \left(\left| \frac{\partial^2}{\partial s^2} u_1(x, s) - \frac{\partial^2}{\partial s^2} u_1(y, s) \right| \right. \\
&\quad \left. + \left| \mu_2 u_1 \left(\frac{\partial^2}{\partial s^2} u_1(x, s) + \mu_3 u_1(x, s), s \right) \right. \right. \\
&\quad \left. \left. - \mu_2 u_1 \left(\frac{\partial^2}{\partial s^2} u_1(y, s) + \mu_3 u_1(y, s), s \right) \right| \right) ds d\tau \\
&\leq L_0(t)|x - y| + \int_0^t \int_0^\tau L_1(s) \left(L_0^3(s) |x - y| \right. \\
&\quad \left. + L_1(s) \left(L_0^3(s) |x - y| + L_1(s) |x - y| \right) \right) ds d\tau \\
&\leq \left(L_0(t) + \int_0^t \int_0^\tau \left(\left(L_1(s) + L_1^2(s) \right) C_0(s) + L_1^3(s) \right) ds d\tau \right) |x - y| \\
&:= \left(L_0(t) + \int_0^t \int_0^\tau C_1(s) ds d\tau \right) |x - y| := L_2(t) |x - y|,
\end{aligned} \tag{2.15}$$

where

$$L_2(t) := L_0(t) + \int_0^t \int_0^\tau C_1(s) ds d\tau,$$

$$C_1(t) := \left(L_1(t) + L_1^2(t) \right) C_0(t) + L_1^3(t).$$

Moreover

$$\begin{aligned} & \left| \frac{\partial^2}{\partial t^2} u_2(x, t) - \frac{\partial^2}{\partial t^2} u_2(y, t) \right| \\ & \leq L_1(t) \left(\left| \frac{\partial^2}{\partial t^2} u_1(x, t) - \frac{\partial^2}{\partial t^2} u_1(y, t) \right| + \left| \mu_2 u_1 \left(\frac{\partial^2}{\partial t^2} u_1(x, t) \right. \right. \right. \\ & \quad \left. \left. \left. + \mu_3 u_1(x, t), t \right) - \mu_2 u_1 \left(\frac{\partial^2}{\partial t^2} u_1(y, t) + \mu_3 u_1(y, t), t \right) \right| \right) \\ & \leq L_1(t) \left(L_0^3(t) |x - y| + L_1(t) \left(\left| \frac{\partial^2}{\partial t^2} u_1(x, t) - \frac{\partial^2}{\partial t^2} u_1(y, t) \right| \right. \right. \\ & \quad \left. \left. + |\mu_3 u_1(x, t) - \mu_3 u_1(y, t)| \right) \right) \\ & \leq \left(\left(L_1(t) + L_1^2(t) \right) L_0^3(t) + L_1^3(t) \right) |x - y| \\ & = \left(\left(L_1(t) + L_1^2(t) \right) C_0(t) + L_1^3(t) \right) |x - y| := C_1(t) |x - y|. \end{aligned}$$

Repeating the previous calculation for u_3 we get

$$\begin{aligned} & |u_3(x, t) - u_3(y, t)| \\ & \leq L_0(t) |x - y| + \int_0^t \int_0^\tau L_2(s) \left(\left| \frac{\partial^2}{\partial s^2} u_2(x, s) \frac{\partial^2}{\partial s^2} u_2(y, s) \right| \right. \\ & \quad \left. + \left| \mu_2 u_2 \left(\frac{\partial^2}{\partial s^2} u_2(x, s) + \mu_3 u_2(x, s), s \right) \right. \right. \\ & \quad \left. \left. - \mu_2 u_2 \left(\frac{\partial^2}{\partial s^2} u_2(y, s) + \mu_3 u_2(y, s), s \right) \right| \right) ds d\tau \quad (2.16) \\ & \leq \left(L_0(t) + \int_0^t \int_0^\tau \left(\left(L_2(s) + L_2^2(s) \right) C_1(s) + L_2^3(s) \right) ds d\tau \right) |x - y| \\ & := \left(L_0(t) + \int_0^t \int_0^\tau C_2(s) ds d\tau \right) |x - y| := L_3(t) |x - y|, \end{aligned}$$

where

$$L_3(t) := L_0(t) + \int_0^t \int_0^\tau C_2(s) ds d\tau,$$

and

$$C_2(t) := \left(L_2(t) + L_2^2(t) \right) C_1(t) + L_2^3(t).$$

We have also

$$\left| \frac{\partial^2}{\partial t^2} u_3(x, t) - \frac{\partial^2}{\partial t^2} u_3(y, t) \right| \leq C_2(t) |x - y|.$$

Next, we proceed by induction. Let $L_0(t) := \sigma + t\omega$ and $C_0(t) := L_0^3(t)$,

$$\begin{aligned} C_n(t) &:= \left(L_n(t) + L_n^2(t) \right) C_{n-1}(t) + L_n^3(t) \\ L_n(t) &:= L_0(t) + \int_0^t \int_0^\tau C_{n-1}(s) ds d\tau, \quad n \geq 1. \end{aligned} \quad (2.17)$$

From (2.13) – (2.16), by induction on n , we obtain

$$|u_{n+1}(x, t) - u_{n+1}(y, t)| \leq L_{n+1}(t) |x - y|, \quad (2.18)$$

$$\left| \frac{\partial^2}{\partial t^2} u_{n+1}(x, t) - \frac{\partial^2}{\partial t^2} u_{n+1}(y, t) \right| \leq C_n(t) |x - y|. \quad (2.19)$$

We introduce a definition. We call (v_n) a stationary sequence in x if

$$|v_{n+1}(x, t) - v_n(x, t)| \leq f_n(t),$$

where (f_n) is a non-negative sequence of real function defined on $[0, T]$. If $f_n = f$ for all n , we say that (v_n) is uniformly stationary sequence in x .

Step 4: (u_n) and $(\frac{\partial^2}{\partial t^2} u_n)$ are stationary sequence in x . Direct calculations show that

$$\begin{aligned} |u_1(x, t) - u_0(x, t)| &= \int_0^t \int_0^\tau \left| \mu_1 u_0 \left(\mu_2 u_0 (\mu_3 u_0(x, s), s), s \right) \right| ds d\tau \\ &\leq \int_0^t \int_0^\tau \left(\|p\|_{L^\infty} + t \|q\|_{L^\infty} \right) ds d\tau \\ &= \frac{t^2}{2} \|p\|_{L^\infty} + \frac{t^3}{6} \|q\|_{L^\infty} := A_1(t). \end{aligned} \quad (2.20)$$

$$\begin{aligned} \left| \frac{\partial^2}{\partial t^2} u_1(x, t) - \frac{\partial^2}{\partial t^2} u_0(x, t) \right| &= \left| \mu_1 u_0 \left(\mu_2 u_0 (\mu_3 u_0(x, s), s), s \right) \right| \\ &\leq \|p\|_{L^\infty} + t \|q\|_{L^\infty} := B_1(t). \end{aligned} \quad (2.21)$$

From (2.20) and (2.21), we deduce

$$A_1(t) := \int_0^t \int_0^\tau B_1(s) ds d\tau. \quad (2.22)$$

$$\begin{aligned} & |u_2(x, t) - u_1(x, t)| \\ & \leq \int_0^t \int_0^\tau \left(A_1(s) + L_0(s) \left(\left| \frac{\partial^2}{\partial s^2} u_1(x, s) - \frac{\partial^2}{\partial s^2} u_0(x, s) \right| \right. \right. \\ & \quad \left. \left. + \left| \mu_2 u_0 \left(\frac{\partial^2}{\partial s^2} u_1(x, s) + \mu_3 u_1(x, s), s \right) \right. \right. \right. \\ & \quad \left. \left. \left. - \mu_2 u_0 \left(\frac{\partial^2}{\partial s^2} u_0(x, s) + \mu_3 u_0(x, s), s \right) \right| \right) \right) ds d\tau \\ & \leq \int_0^t \int_0^\tau \left(\left(1 + L_0(s) + L_0^2(s) \right) A_1(s) + \left(L_0(s) + L_0^2(s) \right) B_1(s) \right) ds d\tau \\ & := A_2(t). \end{aligned} \quad (2.23)$$

$$\begin{aligned} & \left| \frac{\partial^2}{\partial t^2} u_2(x, t) - \frac{\partial^2}{\partial t^2} u_1(x, t) \right| \\ & \leq A_1(t) + L_0(t) \left(\left| \frac{\partial^2}{\partial t^2} u_1(x, t) - \frac{\partial^2}{\partial t^2} u_0(x, t) \right| + \left| \mu_2 u_0 \left(\frac{\partial^2}{\partial t^2} u_1(x, t) \right. \right. \right. \\ & \quad \left. \left. \left. + \mu_3 u_1(x, t), t \right) - \mu_2 u_0 \left(\frac{\partial^2}{\partial t^2} u_0(x, t) + \mu_3 u_0(x, t), t \right) \right| \right) \\ & \leq A_1(t) \left(1 + L_0(t) + L_0^2(t) \right) + \left(L_0(t) + L_0^2(t) \right) B_1(t) \\ & := B_2(t). \end{aligned} \quad (2.24)$$

Combining (2.23) and (2.24) gives

$$A_2(t) := \int_0^t \int_0^\tau B_2(s) ds d\tau. \quad (2.25)$$

From (2.20) and (2.23), by inducting on n , we derive

$$|u_{n+1}(x, t) - u_n(x, t)| \leq A_{n+1}(t) \quad (2.26)$$

and

$$\left| \frac{\partial^2}{\partial t^2} u_{n+1}(x, t) - \frac{\partial^2}{\partial t^2} u_n(x, t) \right| \leq B_{n+1}(t), \quad (2.27)$$

where

$$\begin{aligned} A_{n+1}(t) &:= \int_0^t \int_0^\tau B_{n+1}(s) ds d\tau, \\ B_{n+1}(t) &:= \left(1 + L_{n-1}(t) + L_{n-1}^2(t)\right) A_n(t) \\ &\quad + \left(L_{n-1}(t) + L_{n-1}^2(t)\right) B_n(t), \quad n \geq 1. \end{aligned} \quad (2.28)$$

In the following step, we select T_0 for which we prove also that (u_n) and $(\frac{\partial^2}{\partial t^2} u_n)$ are uniformly stationary sequences.

Step 5: Existence of a local solution. Because $\sigma < 1$, we can find $T_0 > 0$, $0 < M < 1$, $0 < h < 1$ such that for $t \in [0, T_0]$, we have

$$\sigma + t\omega + M\frac{t^2}{2} \leq M < 2M < h; \quad M + 2M^2 \leq 1; \quad 2M + (1 + 2M)\frac{t^2}{2} < h. \quad (2.29)$$

From (2.29) we obtain

$$\begin{aligned} L_0(t) &= \sigma + t\omega \leq M, \\ L_1(t) &\leq \sigma + t\omega + \int_0^t \int_0^\tau M^3 ds d\tau = \sigma + t\omega + M^3 \frac{t^2}{2} \leq M, \\ L_2(t) &\leq \sigma + t\omega + \int_0^t \int_0^\tau \left(M^3 + M^4 + M^5\right) ds d\tau \leq \sigma + t\omega + M\frac{t^2}{2} \leq M, \\ C_0(t) &= L_0^3(t) \leq M^3 \leq M, \\ C_1(t) &\leq \left(M + M^2\right)M + M^3 = M(M + 2M^2) \leq M, \\ C_2(t) &\leq \left(M + M^2\right)M + M^3 = M(M + 2M^2) \leq M. \end{aligned} \quad (2.30)$$

Now, by induction on n , we conclude that

$$\begin{aligned} C_n(t) &\leq M, \\ L_{n+1}(t) &\leq \sigma + t\omega + M\frac{t^2}{2} \leq M. \end{aligned} \quad (2.31)$$

Hence we derive

$$\begin{aligned}
B_2(t) &\leq A_1(t)(1 + M + M^2) + B_1(t)(M + M^2) \\
&\leq (1 + M + M^2) \int_0^t \int_0^\tau B_1(s) ds d\tau + B_1(t)(M + M^2) \\
&\leq \|B_1\|_{L^\infty} \frac{t^2}{2} (1 + M + M^2) + \|B_1\|_{L^\infty} (M + M^2) \\
&\leq \|B_1\|_{L^\infty} \left(\frac{t^2}{2} (1 + 2M) + 2M \right) \\
&\leq \|B_1\|_{L^\infty} h.
\end{aligned} \tag{2.32}$$

From (2.32) we obtain

$$\|B_2\|_{L^\infty} \leq \|B_1\|_{L^\infty} h. \tag{2.33}$$

By a similar argument, we get

$$\begin{aligned}
B_3(t) &\leq \|B_2\|_{L^\infty} \frac{t^2}{2} (1 + M + M^2) + \|B_2\|_{L^\infty} (M + M^2) \\
&\leq \|B_2\|_{L^\infty} \left(\frac{t^2}{2} (1 + 2M) + 2M \right) \leq \|B_2\|_{L^\infty} h.
\end{aligned} \tag{2.34}$$

So

$$\|B_3\|_{L^\infty} \leq \|B_2\|_{L^\infty} h. \tag{2.35}$$

From (2.33) and (2.35), by induction on n , we conclude that

$$\|B_{n+1}\|_{L^\infty} \leq \|B_n\|_{L^\infty} h. \tag{2.36}$$

In addition, from (2.28) we deduce

$$\|A_{n+1}\|_{L^\infty} \leq \|B_{n+1}\|_{L^\infty} \frac{T_0^2}{2}. \tag{2.37}$$

Due to (2.36), we see the series $\sum B_{n+1}(t)$ converges absolutely and uniformly, hence by (2.27) there exists ϕ_∞ such that

$$\frac{\partial^2}{\partial t^2} u_n \rightarrow \phi_\infty \tag{2.38}$$

uniformly in $\mathbb{R} \times [0, T_0]$.

Similarly, from (2.26)) and (2.37), we conclude that $\sum A_{n+1}(t)$ converges absolutely and uniformly and there exists u_∞ such that

$$u_n \rightarrow u_\infty \quad (2.39)$$

uniformly in $\mathbb{R} \times [0, T_*]$.

We remark that $|u_\infty(x, t) - u_\infty(y, t)| \leq M|x - y|$.

Now we are proving that $u_\infty(x, t)$ is a solution of (2.9). It is clear that

$$\begin{aligned} & \left| \mu_1 u_n \left(\frac{\partial^2}{\partial t^2} u_n(x, t) + \mu_2 u_n \left(\frac{\partial^2}{\partial t^2} u_n(x, t) + \mu_3 u_n(x, t), t \right), t \right) \right. \\ & \quad \left. - \mu_1 u_\infty \left(\phi_\infty(x, t) + \mu_2 u_\infty \left(\phi_\infty(x, t) + \mu_3 u_\infty(x, t), t \right), t \right) \right| \\ & \leq \|u_n - u_\infty\|_{L^\infty} + M \left(\left| \frac{\partial^2}{\partial t^2} u_n(x, t) - \phi_\infty(x, t) \right| \right. \\ & \quad \left. + \left| \mu_2 u_n \left(\frac{\partial^2}{\partial t^2} u_n(x, t) + \mu_3 u_n(x, t), t \right) - \mu_2 u_\infty \left(\phi_\infty(x, t) \right. \right. \right. \\ & \quad \left. \left. \left. + \mu_3 u_\infty(x, t), t \right) \right| \right) \\ & \leq \|u_n - u_\infty\|_{L^\infty} (1 + M + M^2) \\ & \quad + \left\| \frac{\partial^2}{\partial t^2} u_n - \phi_\infty \right\|_{L^\infty} (M + M^2) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (2.40)$$

From 2.40, we deduce that

$$\begin{aligned} u_\infty(x, t) = u_0(x, t) + \int_0^t \int_0^\tau & \mu_1 u_\infty \left(\phi_\infty(x, s) + \mu_2 u_\infty \left(\phi_\infty(x, s) \right. \right. \\ & \left. \left. + \mu_3 u_\infty(x, s), s \right), s \right) ds d\tau. \end{aligned} \quad (2.41)$$

Moreover, we have

$$\begin{aligned}
\left| \phi_\infty(x, t) - \frac{\partial^2}{\partial t^2} u_\infty(x, t) \right| &\leq \left\| \phi_\infty - \frac{\partial^2}{\partial t^2} u_n \right\|_{L^\infty} + \left| \frac{\partial^2}{\partial t^2} u_n(x, t) - \frac{\partial^2}{\partial t^2} u_\infty(x, t) \right| \\
&\leq \left\| \phi_\infty - \frac{\partial^2}{\partial t^2} u_n \right\|_{L^\infty} + \|u_{n-1} - u_\infty\|_{L^\infty} + M \left(\left| \frac{\partial^2}{\partial t^2} u_{n-1}(x, t) \right. \right. \\
&\quad \left. \left. - \frac{\partial^2}{\partial t^2} u_\infty(x, t) \right| + \left| \mu_2 u_{n-1} \left(\frac{\partial^2}{\partial t^2} u_{n-1}(x, t) + \mu_3 u_{n-1}(x, t), t \right) \right. \right. \\
&\quad \left. \left. - \mu_2 u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, t) + \mu_3 u_\infty(x, t), t \right) \right| \right) \\
&\leq \left(\left\| \phi_\infty - \frac{\partial^2}{\partial t^2} u_n \right\|_{L^\infty} + \|u_{n-1} - u_\infty\|_{L^\infty} \right) (1 + M + M^2) \\
&\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \tag{2.42}$$

Hence,

$$\begin{aligned}
u_\infty(x, t) &= u_0(x, t) + \int_0^t \int_0^\tau \mu_1 u_\infty \left(\frac{\partial^2}{\partial s^2} u_\infty(x, s) \right. \\
&\quad \left. + \mu_2 u_\infty \left(\frac{\partial^2}{\partial s^2} u_\infty(x, s) + \mu_3 u_\infty(x, s), s \right), s \right) ds d\tau,
\end{aligned} \tag{2.43}$$

for all $x \in \mathbb{R}$, $t \in [0, T_0]$. Then u_∞ is a solution of (2.9) in $\mathbb{R} \times [0, T_0]$.

Step 6: Uniqueness of the local solution u_∞ . We assume that there exists another

lipschitz solution $u_*(x, t)$ of (2.9). Then

$$\begin{aligned}
& |u_*(x, t) - u_\infty(x, t)| \\
& \leq \int_0^t \int_0^\tau \left(\left| u_* \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_* \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_*(x, s), s \right), s \right) \right. \right. \\
& \quad \left. \left. - u_\infty \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_* \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_*(x, s), s \right), s \right) \right| \right. \\
& \quad \left. + \left| u_\infty \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_* \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_*(x, s), s \right), s \right) \right. \right. \\
& \quad \left. \left. - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, s) + u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, s) + u_\infty(x, s), s \right), s \right) \right| \right) ds d\tau \\
& \leq \int_0^t \int_0^\tau \left(\|u_* - u_\infty\|_{L^\infty} + M \left(\left| \frac{\partial^2}{\partial s^2} u_*(x, s) - \frac{\partial^2}{\partial s^2} u_\infty(x, s) \right| \right. \right. \\
& \quad \left. \left. + \left| u_* \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_*(x, s), s \right) - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, s) + u_\infty(x, s), s \right) \right| \right) \right) ds d\tau \\
& \leq \int_0^t \int_0^\tau \left(\|u_* - u_\infty\|_{L^\infty} + M \left(\left| \frac{\partial^2}{\partial s^2} u_*(x, s) - \frac{\partial^2}{\partial s^2} u_\infty(x, s) \right| \right. \right. \\
& \quad \left. \left. + \left| u_* \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_*(x, s), s \right) - u_\infty \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_*(x, s), s \right) \right| \right. \right. \\
& \quad \left. \left. + \left| u_\infty \left(\frac{\partial^2}{\partial t^2} u_*(x, s) + u_*(x, s), s \right) - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, s) + u_\infty(x, s), s \right) \right| \right) \right) ds d\tau \\
& \leq \int_0^t \int_0^\tau \left(\|u_* - u_\infty\|_{L^\infty} + M \left(\left\| \frac{\partial^2}{\partial s^2} u_* - \frac{\partial^2}{\partial s^2} u_\infty \right\|_{L^\infty} + \|u_* - u_\infty\|_{L^\infty} \right. \right. \\
& \quad \left. \left. + M \left(\left\| \frac{\partial^2}{\partial s^2} u_* - \frac{\partial^2}{\partial s^2} u_\infty \right\|_{L^\infty} + \|u_* - u_\infty\|_{L^\infty} \right) \right) \right) ds d\tau \\
& \leq \int_0^t \int_0^\tau \left((1 + M + M^2) \|u_* - u_\infty\|_{L^\infty} \right. \\
& \quad \left. + (M + M^2) \left\| \frac{\partial^2}{\partial s^2} u_* - \frac{\partial^2}{\partial s^2} u_\infty \right\|_{L^\infty} \right) ds d\tau.
\end{aligned} \tag{2.44}$$

Additionally,

$$\begin{aligned}
& \left| \frac{\partial^2}{\partial t^2} u_\star(x, t) - \frac{\partial^2}{\partial t^2} u_\infty(x, t) \right| \\
&= \left| u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right), t \right) \right. \\
&\quad \left. - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, t) + u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, t) + u_\infty(x, t), t \right), t \right) \right| \\
&\leq \left| u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right), t \right) \right. \\
&\quad \left. - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right), t \right) \right| \\
&\quad + \left| u_\infty \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right), t \right) \right. \\
&\quad \left. - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, t) + u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, t) + u_\infty(x, t), t \right), t \right) \right| \\
&\leq \|u_\star - u_\infty\|_{L^\infty} + M \left(\left| \frac{\partial^2}{\partial t^2} u_\star(x, t) - \frac{\partial^2}{\partial t^2} u_\infty(x, t) \right| \right. \\
&\quad \left. + \left| u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right) - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, t) + u_\infty(x, t), t \right) \right| \right) \quad (2.45) \\
&\leq \|u_\star - u_\infty\|_{L^\infty} + M \left(\left\| \frac{\partial^2}{\partial t^2} u_\star - \frac{\partial^2}{\partial t^2} u_\infty \right\|_{L^\infty} \right. \\
&\quad + \left| u_\star \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right) - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right) \right| \\
&\quad + \left| u_\infty \left(\frac{\partial^2}{\partial t^2} u_\star(x, t) + u_\star(x, t), t \right) - u_\infty \left(\frac{\partial^2}{\partial t^2} u_\infty(x, t) + u_\infty(x, t), t \right) \right| \Bigg) \\
&\leq \|u_\star - u_\infty\|_{L^\infty} + M \left(\left\| \frac{\partial^2}{\partial t^2} u_\star - \frac{\partial^2}{\partial t^2} u_\infty \right\|_{L^\infty} + \|u_\star - u_\infty\|_{L^\infty} \right. \\
&\quad \left. + M \left(\left| \frac{\partial^2}{\partial t^2} u_\star(x, t) - \frac{\partial^2}{\partial t^2} u_\infty(x, t) \right| + |u_\star(x, t) - u_\infty(x, t)| \right) \right) \\
&\leq \|u_\star - u_\infty\|_{L^\infty} (1 + M + M^2) \\
&\quad + \left\| \frac{\partial^2}{\partial t^2} u_\star - \frac{\partial^2}{\partial t^2} u_\infty \right\|_{L^\infty} (M + M^2) \\
&\leq (1 + 2M) \|u_\star - u_\infty\|_{L^\infty} + 2M \left\| \frac{\partial^2}{\partial t^2} u_\star - \frac{\partial^2}{\partial t^2} u_\infty \right\|_{L^\infty}.
\end{aligned}$$

From (2.45), we deduce

$$\left\| \frac{\partial^2}{\partial t^2} u_\star - \frac{\partial^2}{\partial t^2} u_\infty \right\|_{L^\infty} \leq \frac{1+2M}{1-2M} \|u_\star - u_\infty\|_{L^\infty}. \quad (2.46)$$

From (2.11), (2.44) and (2.46), we deduce

$$|u_\star(x, t) - u_\infty(x, t)| \leq \left(\frac{1+2M}{1-2M} \right) \frac{T_0^2}{2} \|u_\star - u_\infty\|_{L^\infty}. \quad (2.47)$$

This shows that $u_\infty \equiv u_\star$ and the proof is complete. \square

Remark 2.1. It is clear that a trivial example for problem (1.8) is that $u(x, t) = 0$ is a solution for $p(x) = q(x) = 0$.

Remark 2.2. In this paper, we only consider the existence and uniqueness of a local solution to problem (1.8). It is of course interesting to investigate the behavior of this solution for some special cases of the initial more regular data p and q . We do not think that such problems are trivial.

Remark 2.3. A numerical algorithm for problem (1.8) is still open. We believe that various specific differential equations with self-reference of the general form

$$Au(x, t) = u(Bu(x, t), t),$$

where $A : X \rightarrow \mathbb{R}$ and $B : X \rightarrow \mathbb{R}$ are two functionals, X is a function space, $u = u(x, t)$, $(x, t) \in \mathbb{R} \times [0, +\infty)$ is an unknown function, can be solved numerically.

3 A remark for particular initial data

Now we present a particular situation that show as the initial value are very important in the study the iterative procedure considered in previous section, in particular if we assume $p(x) = p_0$, $q(x) = q_0$; p_0 and q_0 are two given real constants.

Now, suppose $p(x) = p_0$ and $q(x) = q_0$, where p_0 and q_0 are two given real constants. We consider, as in previous section

$$u_0(x, t) = p_0 + tq_0, \quad (3.48)$$

and remark that

$$\begin{aligned}
u_1(x, t) &= u_0(x, t) + \int_0^t \int_0^\tau u_0(u_0(u_0(x, s), s), s) ds d\tau \\
&= u_0(x, t) + \int_0^t \int_0^\tau u_0(u_0(p_0 + sq_0, s), s) ds d\tau \\
&= p_0 + tq_0 + \int_0^t \int_0^\tau (p_0 + sq_0) ds d\tau \\
&= p_0 + tq_0 + p_0 \frac{t^2}{2} + q_0 \frac{t^3}{6} \\
&= p_0 \left(1 + \frac{t^2}{2!}\right) + q_0 \left(t + \frac{t^3}{3!}\right).
\end{aligned} \tag{3.49}$$

Therefore

$$\frac{\partial^2}{\partial t^2} u_1(x, t) = p_0 + tq_0 = u_0(x, t). \tag{3.50}$$

In addition, we get

$$\begin{aligned}
u_2(x, t) &= u_0(x, t) + \int_0^t \int_0^\tau u_1 \left(\frac{\partial^2}{\partial s^2} u_1(x, s) + u_1 \left(\frac{\partial^2}{\partial s^2} u_1(x, s) \right. \right. \\
&\quad \left. \left. + u_1(x, s), s \right), s \right) ds d\tau \\
&= u_0(x, t) + \int_0^t \int_0^\tau \left(p_0 \left(1 + \frac{s^2}{2!}\right) + q_0 \left(s + \frac{s^3}{3!}\right) \right) ds d\tau \\
&= p_0 \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!}\right) + q_0 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!}\right).
\end{aligned} \tag{3.51}$$

Now, by induction on k we obtain

$$u_k(x, t) = p_0 \sum_{i=0}^k \frac{t^{2i}}{(2i)!} + q_0 \sum_{i=0}^k \frac{t^{2i+1}}{(2i+1)!}.$$

We deduce

$$\begin{aligned}
u_{k+1}(x, t) &= u_0(x, t) + \int_0^t \int_0^\tau \left(p_0 \sum_{i=0}^k \frac{s^{2i}}{(2i)!} + q_0 \sum_{i=0}^k \frac{t^{2i+1}}{(2i+1)!} \right) ds d\tau \\
&= p_0 \left(1 + \sum_{i=0}^k \frac{t^{2i+2}}{(2i+2)!}\right) + q_0 \left(t + \sum_{i=0}^k \frac{t^{2i+3}}{(2i+3)!}\right).
\end{aligned} \tag{3.52}$$

From (3.48) – (3.52) we obtain

$$u_n(x, t) = p_0 \sum_{i=0}^n \frac{t^{2i}}{(2i)!} + q_0 \sum_{i=0}^n \frac{t^{2i+1}}{(2i+1)!}, \tag{3.53}$$

$$\frac{\partial^2}{\partial t^2} u_{n+1}(x, t) = u_{n-1}(x, t). \quad (3.54)$$

Letting n go to infinity, for all $t \in [0, T], T > 0$, we get

$$u_*(x, t) = \begin{cases} Ce^t, & p_0 = q_0 = C \\ p_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + q_0 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = p_0 \cosh t + q_0 \sinh t, & p_0 \neq q_0. \end{cases} \quad (3.55)$$

But it is easy to prove that u_* are solution of (1.8). The functions u_* are solution of the ordinary differential equation $\ddot{u}(t) = u(t)$.

Hence we have the following situation. The problem (2.9) generated an integral equation; starting from non-constant initial condition (so that almost one of p, q depend explicitly on x) the iteration procedure give a local solution of problem (2.9). But starting from constant initial condition (p and q together constant) the same iteration procedure give a solution of a different problem. This seem to be a very interesting situation.

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